Sojourn times in finite Markov processes

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Rapports de Recherche

N° 812

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MARS 1988

RR812

IRISA

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Publication Interne n° 395
Février 1988
22 Pages

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Abstract Sojourn times of Markov processes in subsets of the finite state space are considered. We give a closed form of the distribution of the n^{th} sojourn time in a given subset of states. The asymptotical behaviour of this distribution when time goes to infinity is analyzed, both in the discrete and the continuous time cases. We consider the usually pseudo-aggregated Markov process canonically constructed from the previous one by collapsing the states of each subset of the partition. The relation between limits of moments of the sojourn time distributions in the original Markov process and the moments of the corresponding holding times of the pseudo-aggregated one is also studied.

MARKOV PROCESSES, AGGREGATION, SOJOURN TIMES, TRANSIENT ANALYSIS, PERFORMANCE EVALUATION, RELIABILITY MODELING

Temps de Séjour dans les Processus Markoviens Finis

Résumé On considère les temps de séjour dans des sous-ensembles d'états de processus Markoviens à espace d'état fini. On donne la distribution du $n^{ième}$ temps de séjour dans un sous-ensemble d'états donné. Le comportement asymptotique de cette distribution est étudié dans les cas discret et continu. On s'intéresse aussi au processus pseudo-agregé construit à partir de l'originel par agrégation des états suivant une partition donnée. Les relations entre les limites des moments succéssifs des temps de séjour sur le processus de départ et les temps de séjour correspondant sur le processus pseudo-agregé sont aussi étudiées.

PROCESSUS MARKOVIENS, AGREGATION, TEMPS DE SEJOUR, REGIME TRANSITOIRE, EVALUATION DE PERFORMANCES, MODELES DE FIABILITE

1 Introduction

Consider an homogeneous irreducible Markov process X evolving in discrete or continuous time; in the first case we will speak about a chain. We can distinguish a subset of states and then consider the random variable "time spent by process X in the given subset of the state space in its n^{th} visit to the subset". This kind of subject is of interest in areas such as reliability or performability, when a subset of the state space corresponds, for instance, to a fixed level of performance of the system. Since the processes considered here are irreducible, in such a context this means that we are concerned by systems that have always recovery procedures, that is, models without absorbing classes. The analysis of these random variables is the main topic of this note.

The *sojourn* time of a process X in a subset of states will be an integer valued random variable if X is a chain or a real valued one in the case of a continuous time process. The distributions of these random variables are given in Sections 2 and 3 for the discrete time and continuous time cases respectively. We also study the behaviour of these distributions when time goes to infinity, that is, for $n \to +\infty$ when considering the n^{th} visit to the the given subset.

Let $E = \{1, 2, ..., N\}$ be the finite state space of the given process and let $\mathcal{B} = \{B(1), B(2), ..., B(M)\}$ be a partition of E. Let F be the set of integers $\{1, 2, ..., M\}$ and define the mapping $T : \mathbb{R}^N \to \mathbb{R}^M$ by:

$$T.v = w \qquad \text{where } \; w(m) = \sum_{i \in B(m)} v(i)$$

To the given process X we associate the aggregated stochastic process Y with values on F, defined by:

$$Y_t = m \iff X_t \in B(m) \;\text{ for all values of } t \; (t \in \mathbb{N} \text{ or } t \in \mathbb{R}).$$

We easily deduce from this definition and the irreducibility of X that the process Y obtained is also irreducible but it need not be Markov, not even homogeneous. In [1] and in [2], we can find conditions under which the aggregated chain Y (i.e. in the discrete time case) is also Markov homogeneous. Anyway, we can construct an homogeneous Markov process Z on the state space F canonically associated to the given process X which will be called the *pseudo-aggregated* process of X with respect to the partition $\mathcal{B}$. This pseudo-aggregation is briefly presented in Section 4 and the relation between the holding times of Z and the sojourn times in X when we consider the state of Z corresponding to the subset of states of X is analyzed.

In the last section some conclusions are presented.

2 Sojourn times in the discrete time case

Let $X = (X_n)_{n\geq 0}$ be an homogeneous irreducible Markov chain with transition probability matrix P. Denote by x its equilibrium probability distribution, that is: $x = xP$, $x > 0$ and $x1^T = 1$, where 1 denotes a row vector with all the entries equal to the scalar 1, the dimension being defined by the context. Let B be a proper subset of the state space E and denote by B^c the complementary subset $E - B$. We assume for simplicity that $B = \{1, 2, \ldots, L\}$, $1 \leq L < N$. The partition $\{B, B^c\}$ of E induce the following decomposition of P in four submatrices and of α and x in two subvectors:

$$P = \begin{pmatrix} P_B & P_{BB^c} \\ P_{B^cB} & P_{B^c} \end{pmatrix}$$

$$\alpha = \begin{pmatrix} \alpha_B & \alpha_{B^c} \end{pmatrix} \qquad x = \begin{pmatrix} x_B & x_{B^c} \end{pmatrix}$$

We will need the two following elementary lemmas:

Lemma 2.1 *The matrix $I - P_B$ is invertible.*

Proof. Consider the Markov chain on the state space $B \cup \{0\}$ with transition probability matrix P':

$$\begin{array}{rcll} P'(i,j) & = & P_B(i,j) & \forall\, i,j \in B \\ P'(0,j) & = & 0 & \forall\, j \in B \\ P'(i,0) & = & 1 - \sum_{j\in B} P_B(i,j) & \forall\, i \in B \\ P'(0,0) & = & 1 & \end{array}$$

P being irreducible, this chain has exactly two classes: B transient and $\{0\}$ absorbing. Therefore:

$$\lim_{k\to+\infty} P_B^k(i,j) = 0 \quad \forall i,j \in B$$

which implies that $I - P_B$ is invertible. □

In the same way, the matrix $I - P_{B^c}$ is invertible.

Lemma 2.2 *The vector x_B is a solution to the following system:*

$$\begin{cases} \phi \quad \textit{L-dimensioned row vector} \\ \phi = \phi U_B, \quad \phi > 0 \end{cases}$$

$$\textit{where } U_B = P_{BB^c}(I - P_{B^c})^{-1} P_{B^cB} (I - P_B)^{-1}$$

Proof. The result follows immediatly from the following decomposition of the system $x = xP$:

$$\begin{array}{rcrcl} x_B & = & x_B P_B & + & x_{B^c} P_{B^c B} \\ x_{B^c} & = & x_B P_{BB^c} & + & x_{B^c} P_{B^c} \end{array}$$

by replacing in the first equation the value of the vector x_{B^c} obtained from the second one. □

Remark that in the same way, we have:

$$x_{B^c} = x_{B^c} U_{B^c} \quad \text{where } U_{B^c} = P_{B^c B}(I - P_B)^{-1} P_{BB^c}(I - P_{B^c})^{-1}$$

Definition 2.3 *We call "sojourn of X in B" every sequence $X_m, X_{m+1}, ..., X_{m+k}$ where: $k \geq 1$, $X_m, X_{m+1}, ..., X_{m+k-1} \in B$, $X_{m+k} \notin B$ and if $m > 0$, $X_{m-1} \notin B$. This sojourn begins at time m and finishes at time $m + k$. It lasts k.*

Let V_n, $n \geq 1$ be the random variable "state of B in which the n^{th} sojourn of X begins". The hypothesis of irreducibility of the Markov chain X assures the existence of an infinity of sojourns of X in B with probability 1. It is immediat to verify that $(V_n)_{n\geq 1}$ is an homogeneous Markov chain on the state space B. Let G be the $L \times L$ transition probability matrix of this chain and v_n its probability distribution vector after the n^{th} transition: $v_n = (P(V_n = 1), ..., P(V_n = L))$. We have obvioulsy $v_n = v_1 G^{n-1}$. $(V_n)_{n\geq 1}$ is characterized by G and v_1 which are given in the following theorem.

Theorem 2.4 *Matrix G and vector v_1 are given by the following expressions:*

$$\begin{array}{rcl} (i)\ G & = & (I - P_B)^{-1} P_{BB^c}(I - P_{B^c})^{-1} P_{B^c B} \;=\; (I - P_B)^{-1} U_B (I - P_B) \\ (ii)\ v_1 & = & \alpha_B + \alpha_{B^c}(I - P_{B^c})^{-1} P_{B^c B} \end{array}$$

Proof.

(i) Let $i \in B^c$ and $j \in B$. We define $H(i,j) \stackrel{\text{def}}{=} P(V_1 = j / X_0 = i)$ and let H be the $(N - L) \times L$ matrix with entries $H(i,j)$. It is immediat to verify from the Markov properties of X that we have:

for $i \in B^c$ and $j \in B$:

$$P(V_1 = j / X_0 = i) = P(i,j) + \sum_{k \in B^c} P(i,k) P(V_1 = j / X_0 = i, X_1 = k)$$

$$= P(i,j) + \sum_{k \in B^c} P(i,k) P(V_1 = j / X_0 = k)$$

and if $i \in B$ and $j \in B$:

$$G(i,j) = P(V_2 = j/V_1 = i) = P(V_2 = j/X_0 = i)$$
$$= \sum_{k\in E} P(i,k)P(V_2 = j/X_0 = i, X_1 = k)$$
$$= \sum_{k\in B} P(i,k)P(V_2 = j/X_0 = k) + \sum_{k\in B^c} P(i,k)P(V_1 = j/X_0 = k)$$

which gives in matrix notation:

$$H = P_{B^cB} + P_{B^c}H$$
$$G = P_BG + P_{BB^c}H$$

therefore:

$$H = (I - P_{B^c})^{-1}P_{B^cB}$$

and

$$G = (I - P_B)^{-1}P_{BB^c}(I - P_{B^c})^{-1}P_{B^cB}$$

(ii) Let $j \in B$. We have:

$$P(V_1 = j) = P(X_0 = j) + \sum_{i\in B^c} P(V_1 = j/X_0 = i)P(X_0 = i)$$

which can be written in matrix notation:

$$v_1 = \alpha_B + \alpha_{B^c}H$$

and this concludes the proof. □

The chain $(V_n)_{n\geq 1}$ contains only one recurrent class, the set B' of the states of B directly accessible from B^c: $B' = \{j \in B/\exists\ i\ \in\ B^c,\ P(i,j) > 0\}$. Without any loss of generality, we will consider that $B' = \{1, ..., L'\}$ where $1 \leq L' \leq L$. We then denote by $B"$ the set $B - B'$. The partition $\{B', B"\}$ induce the following decomposition on matrices G and H:

$$G = \begin{pmatrix} G' & 0 \\ G" & 0 \end{pmatrix} \qquad H = (H'\ \ 0)$$

In the same way, the partition $\{B', B", B^c\}$ induce the following decomposition on P:

$$P = \begin{pmatrix} P_{B'} & P_{B'B"} & P_{B'B^c} \\ P_{B"B'} & P_{B"} & P_{B"B^c} \\ P_{B^cB'} & 0 & P_{B^c} \end{pmatrix}$$

Now, as for G, we have the following expression for G':

Theorem 2.5

$$\begin{aligned} G' &= (I - P_{B'} - P_{B'B"}(I - P_{B"})^{-1}P_{B"B'})^{-1} \\ &\quad (P_{B'B^c} + P_{B'B"}(I - P_{B"})^{-1}P_{B"B^c})(I - P_{B^c})^{-1}P_{B^cB'} \end{aligned}$$

Proof. The proof is as in (i) of Theorem 2.4. Therefore, we omit it. We give only the system satisfied by matrices G', $G"$ and H':

$$\begin{aligned} G' &= P_{B'}G' + P_{B'B"}G" + P_{B'B^c}H' \\ G" &= P_{B"B'}G' + P_{B"}G" + P_{B"B^c}H' \\ H' &= P_{B^cB'}G' + P_{B^c}H' \end{aligned}$$

By inferring G' from this system we obtain the given expression □

Note that the expression given in the previous theorem can reduce considerably the time necessary to compute G' when $L' < L$: instead of inverting a matrix of size L when computing G from the formula of Theorem 2.4, we have here to invert two matrices of sizes L' and $L - L'$.

We denote now by $H_{B,n}$ for $n \geq 1$ the random variable taking values in $\mathbb{N}^*$: "time spent by X during its n^{th} sojourn in B". We have the following explicit expression for the distribution of $H_{B,n}$.

Theorem 2.6

$$\forall n \in \mathbb{N}^*, \forall k \in \mathbb{N}^*, \quad \mathrm{P}(\mathrm{H}_{\mathrm{B,n}} = \mathrm{k}) = \mathrm{v_n P_B^{k-1}(I - P_B)1^T}$$

Proof. First, we are going to derive the distribution of $H_{B,1}$. Conditionning on the state in which the sojourn in B begins, we obtain:

$$\forall i \in B, \quad P(H_{B,1} = 1/V_1 = i) = \sum_{j \in B^c} P(i,j)$$

Let $k > 1$ and $i \in B$.

$$P(H_{B,1} = k/V_1 = i) = \sum_{j \in B} P(i,j)P(H_{B,1} = k-1/V_1 = j)$$

If we define $h_k \stackrel{\text{def}}{=} (P(H_{B,1} = k/V_1 = 1), \ldots, P(H_{B,1} = k/V_1 = L))$, we can rewrite these two relations as follows:

$$h_1^T = P_{BB^c}1^T = (I - P_B)1^T$$

and for $k > 1$:

$$h_k^T = P_B h_{k-1}^T$$

that is:

$$h_k^T = P_B^{k-1}(I - P_B)1^T \quad \forall k \geq 1$$

and

$$P(H_{B,1} = k) = v_1 P_B^{k-1}(I - P_B)1^T \quad \forall k \geq 1$$

If we consider now the n^{th} sojourn of X in B, we have:

$$\begin{aligned} P(H_{B,n} = k) &= \textstyle\sum_{i \in B} P(H_{B,n} = k/V_n = i)P(V_n = i) \\ &= \textstyle\sum_{i \in B} P(H_{B,1} = k/V_1 = i)P(V_n = i) \\ &= v_n h_k^T \end{aligned}$$

which is the explicit expression given in the statement. □

Let us compute now the moments of the random variable $H_{B,n}$ $(n \geq 1)$. An elementary matrix calculus gives:

$$\begin{aligned} E(H_{B,n}) &= v_n(\textstyle\sum_{k=1}^{+\infty} k P_B^{k-1})(I - P_B)1^T \\ &= v_n(I - P_B)^{-2}(I - P_B)1^T \\ &= v_n(I - P_B)^{-1}1^T \end{aligned}$$

For higher order moments, it is more comfortable to work with factorial moments instead of standard moments. Recall that the k-order factorial moment of a discrete random variable Y, which will be denoted by $FM_k(Y)$, is defined by:

$$FM_k(Y) \stackrel{\text{def}}{=} E(Y(Y-1)...(Y-k+1))$$

The following property holds: if $E(Y^j) = E(Z^j)$ for $j = 1, 2, ..., k-1$ then

$$[\ FM_k(Y) = FM_k(Z) \Longleftrightarrow E(Y^k) = E(Z^k)\ ]$$

See also that $FM_1(Y) = E(Y)$.

Then, by a classical matrix computation as we did for the mean value of $H_{B,n}$, we have:

$$FM_k(H_{B,n}) = k! v_n P_B^{k-1}(I - P_B)^{-k}1^T \tag{1}$$

Let us consider now the asymptotical behaviour of the n^{th} sojourn time of X in B.

Corollary 2.7 *For any $k \geq 1$, the sequence $(P(H_{B,n}=k))_{n\geq 1}$ converges in the sense of Cesaro as $n \to \infty$ and the limit is $vP_B^{k-1}(I-P_B)1^T$ where the L-dimensional row vector v is the unique solution to the following system:*

$$\begin{cases} \phi \quad L-\textit{dimensioned row vector} \\ \phi = \phi G \,,\ \phi \geq 0 \,,\ \phi 1^T = 1 \,; \end{cases}$$

Vector v is given by: $v = \frac{1}{K} x_{B^c} P_{B^c B}$ where K is the normalizing constant $x_{B^c} P_{B^c B} 1^T$. The convergence is simple for any initial distribution of X if and only if G' is aperiodic.

Proof. The proof is an elementary consequence of general properties of Markov chains, since the sequence depends on n only through v_n and $(V_n)_{n\geq 1}$ is a finite homogeneous Markov chain with only one recurrent class B'. See that $v = (v' \ 0)$ according to the partition $\{B', B''\}$ of B. If we consider the block decomposition of G with respect to the partition $\{B', B"\}$, a simple recurrence on integer n allows us to write:

$$G^n = \begin{pmatrix} G'^n & 0 \\ G"G'^{n-1} & 0 \end{pmatrix}$$

G'^n converges in the sense of Cesaro to $1^T v'$ and in the same way, $G"G'^{n-1}$ converges in the sense of Cesaro to $1^T v'$ because $G"1^T = 1^T$ (recall that the dimension of vector 1 is given by the context). The expression for v is easily checked by using Lemma 2.2. The second part of the proof is an immediat corollary of the convergence properties of Markov chains. □

Let us define the random variable $H_{B,\infty}$ with values in $\mathbb{N}^*$ by making its distribution equal to the previous limit:

$$P(H_{B,\infty}=k) = vP_B^{k-1}(I-P_B)1^T \quad \text{for any } k \geq 1$$

Then, by taking Cesaro limits in expression (1) giving the factorial moments of $H_{B,n}$, we obtain the limit given below which needs no proof.

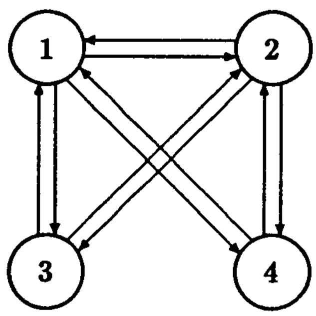
Corollary 2.8 *The sequence $(FM_k(H_{B,n}))_{n\geq 1}$ converges in the sense of Cesaro and we have:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=1}^{n} FM_k(H_{B,l}) = FM_k(H_{B,\infty}) = k! vP_B^{k-1}(I-P_B)^{-k} 1^T$$

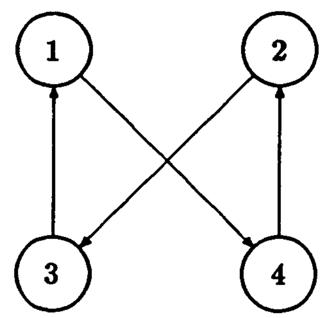
The convergence is simple for any initial distribution of X if and only if the matrix G' is aperiodic.

Remark: There is no relation between the periodicity of P and the periodicity of G'. That is, the four situations obtained by combination of the two properties *periodicity* and *aperiodicity* of each matrix are possible as we show below in four examples. $E = \{1,2,3,4\}$, $B = \{1,2\}$ and there is an arrow between two states if and only if the corresponding transition probability is strictly positive.

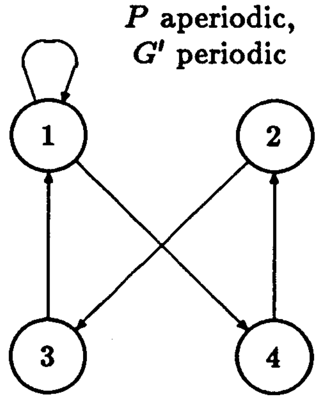
P and G' aperiodic



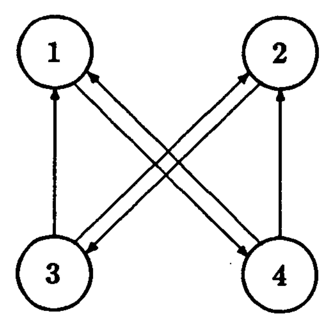
P and G' periodic



P aperiodic,
G' periodic



P periodic,
G' aperiodic



Example. Let us illustrate this results with an example. We consider an homogeneous Markov chain with state space $\{1,2,3,4\}$ given by the following transition probability matrix:

$$P = \left(\begin{array}{cc|cc} 0 & 1/4 & 1/4 & 1/2 \\ 1/2 & 0 & 1/4 & 1/4 \\ \hline 1/4 & 3/4 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{array} \right)$$

The selected subset of states is $B = \{1,2\}$. The stationary distribution vector is $x = \frac{1}{13}\begin{pmatrix} 4 & 4 & 2 & 3 \end{pmatrix}$ and we take $\alpha = x$, that is, we consider the chain in steady state. To compute the distribution of $H_{B,n}$ let us give the following intermediate values:

$$H = P_{B^cB} = \frac{1}{4}\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \quad \text{since } P_{B^c} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$v_1 = \frac{1}{13}\begin{pmatrix} 6 & 7 \end{pmatrix}$$

$$G = \frac{1}{56}\begin{pmatrix} 23 & 33 \\ 22 & 34 \end{pmatrix}$$

$$v_2 = v_1 G = \frac{1}{182}\begin{pmatrix} 73 & 109 \end{pmatrix}$$

$$v = \frac{1}{K} x_{B^c} P_{B^cB} = \frac{1}{5}\begin{pmatrix} 2 & 3 \end{pmatrix}$$

$$P_B^{2j} = \left(\frac{1}{8}\right)^j I \quad \text{for } j \geq 0 \quad \text{where } I \text{ is the 2-order identity matrix}$$

$$P_B^{2m+1} = \frac{1}{4}\left(\frac{1}{8}\right)^m \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \quad \text{for } m \geq 0$$

A classical matrix computation gives:

$$G^n = \frac{1}{5}\begin{pmatrix} 2 + 3 \times 56^{-n} & 3 - 3 \times 56^{-n} \\ 2 - 2 \times 56^{-n} & 3 + 2 \times 56^{-n} \end{pmatrix}$$

Then, after some algebra we get:

$$\begin{aligned} P(H_{B,n} = 2h) &= v_1 G^{n-1} P_B^{2h-1} (I - P_B) 1^T \\ &= \left(\frac{1}{8}\right)^h \frac{143 \times 56^n - 448}{65 \times 56^n} \quad \text{for } h \geq 1 \end{aligned}$$

$$\begin{aligned} P(H_{B,n} = 2h + 1) &= v_1 G^{n-1} P_B^{2h} (I - P_B) 1^T \\ &= \frac{1}{4}\left(\frac{1}{8}\right)^h \frac{156 \times 56^n + 224}{65 \times 56^n} \quad \text{for } h \geq 0 \end{aligned}$$

3 Sojourn times in the continuous time case

Let now X$=(X_t)_{t\geq 0}$ be an irreducible homogeneous Markov process on the state space $E = \{1, ..., N\}$. Let A be the infinitesimal generator of this process, where $A(i,i) \stackrel{\text{def}}{=} -\sum_{j\neq i} A(i,j)$ and define $\lambda_i \stackrel{\text{def}}{=} -A(i,i)\ \forall i \in E$. That is, λ_i represent the output rate of state i. We will denote by Λ the diagonal matrix whose i^{th} entry is λ_i. We will denote also respectively by α and π the initial and the stationary distributions of the process X. The transition probability matrix of the embedded discrete time Markov chain at the instants of state change is: $P = I + \Lambda^{-1}A$ and its stationary distribution is: $x = \pi\Lambda/\pi\Lambda 1^T$.

As in the discrete time case, we consider a subset B of E and we conserve the notations for B, B', $B"$, for the decomposion of P and the decomposion of A with respect to the partitions $\{B, B^c\}$ or $\{B', B", B^c\}$. A sojourn of X in B is now a sequence $X(t_m), ..., X(t_{m+k})$, $k \geq 1$, where the t_i are instants of transition, $X(t_m), ..., X(t_{m+k-1}) \in B$, $X(t_{m+k}) \notin B$ and if $m > 0$, $X(t_{m-1}) \notin B$. This sojourn begins at time t_m and finishes at time t_{m+k}. It lasts $t_{m+k} - t_m$.

Let $H_{B,n}$ be the random variable "time spent during the n^{th} sojourn of X in B". The hypothesis of irreducibility of the Markov process X assures the existence of an infinity of sojourns of X in B with probability 1. Defining V_n as in the previous case for the embedded discrete time Markov chain, we have:

Theorem 3.1

$$P(H_{B,n} \leq t) = 1 - v_n e^{A_B t} 1^T$$

where v_n is explicited in Theorem 2.4.

Proof. Let us construct an homogeneous Markov process $Y = (Y_t)_{t\geq 0}$ on the state space $B \cup \{a\}$ where a is an absorbing state and the matrix A' is its infinitesimal generator:

$$A' \stackrel{\text{def}}{=} \begin{pmatrix} A_B & A_{Ba} \\ 0 & 0 \end{pmatrix}$$

A_{Ba} is the column vector defined by $A_{Ba}(i) \stackrel{\text{def}}{=} \sum_{j\in B^c} A(i,j)$, $i \in B$.

Therefore, we have trivially:

$$\forall i \in B, \quad P_i(H_{B,1} \leq t) = P_i(Y_t = a) = 1 - \sum_{j\in B} e^{A_B t}(i,j)$$

where $P_i(.) \stackrel{\text{def}}{=} P(. / X_0 = i)$.

$\forall i \in B^c$, the following relation holds:

$$P_i(H_{B,1} \leq t) = \sum_{j\in E} P(i,j) P_j(H_{B,1} \leq t)$$

These two relations give:

$$P(H_{B,1} \leq t) = 1 - (\alpha_B + \alpha_{B^c}(I - P_{B^c})^{-1} P_{B^c B}) e^{A_B t} 1^T$$

So:

$$P(H_{B,1} \le t) = 1 - v_1 e^{A_B t} 1^T$$

For any $i \in B$ and for any $n \ge 1$, we can write:

$$P(H_{B,n} \le t / V_n = i) = P_i(H_{B,1} \le t)$$

We deduce:

$$\begin{aligned} P(H_{B,n} \le t) &= \sum_{i \in B} P(H_{B,n} \le t / V_n = i) P(V_n = i) \\ &= \sum_{i \in B} P(V_n = i) P_i(H_{B,1} \le t) \\ &= v_n(1^T - e^{A_B t} 1^T) \\ &= 1 - v_n e^{A_B t} 1^T \end{aligned}$$

which is the explicit form of the distribution of $H_{B,n}$. $\square$

The moments of $H_{B,n}$ are then easily derived. We have:

$$E(H_{B,n}^k) = (-1)^k k! v_n A_B^{-k} 1^T \quad \text{for any } k \ge 1$$

The analogous of Corollary 2.7 is the following:

Corollary 3.2 *The sequence $(P(H_{B,n} \le t))_{n \ge 1}$ converges in the sense of Cesaro as $n \to \infty$ and the limit is $1 - v e^{A_B t} 1^T$ where the L-dimensioned row vector v is the stationary distribution of the Markov chain $(V_n)_{n \ge 1}$ which has been given in Corollary 2.7. The convergence is simple for any initial distribution of X if and only if G' is aperiodic.*

Proof. Note that $A_B = -\Lambda_B(I - P_B)$ and that the canonical embedded Markov chain with transition probability matrix P is irreducible. This implies that the matrix $(I - P_B)^{-1}$ exists (Lemma 2.1) and so A_B^{-1} exists too. The proof of the corollary is then as in Corollary 2.7. $\square$

As in the discrete time case, we define the positive real valued random variable $H_{B,\infty}$ by: $P(H_{B,\infty} \le t) \stackrel{\text{def}}{=} 1 - v e^{A_B t} 1^T$. We can derive then the Cesaro limits of the k-order moments of $H_{B,n}$ and easily verify the following relation:

Corollary 3.3 *For any $k \ge 1$, the sequence $\left(E(H_{B,n}^k)\right)_{n \ge 1}$ converges in the sense of Cesaro and we have:*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} E(H_{B,l}^k) = E(H_{B,\infty}^k) = (-1)^k k! v A_B^{-k} 1^T$$

The convergence is simple for any initial distribution of X if and only if the matrix G' is aperiodic.

4 Pseudo-aggregation

4.1 The pseudo-aggregated process

We consider first the discrete time case. Let $X = (X_n)_{n\geq 0}$ be an homogeneous irreducible Markov chain with transition probability matrix P and equilibrium probability distribution x. We construct the pseudo-aggregated homogeneous Markov chain $Z = (Z_n)_{n\geq 0}$ from X with respect to the partition $\mathcal{B}$ by defining its transition probability matrix $\hat{P}$ as follows:

$$\forall i,j \in F,\ \ \hat{P}(i,j) \stackrel{\text{def}}{=} \sum_{k\in B(i)} \frac{x(k)}{\sum_{h\in B(i)} x(h)} \sum_{l\in B(j)} P(k,l)$$

If we denote by α the initial distribution of X, the initial distribution of Z is $T.\alpha$ where the operator T has been defined in Section 2 (page 2). If the initial distribution α leads to an homogeneous Markov chain for Y, then Z and Y define the same homogeneous Markov chain (see [1] or [2]). It is immediat that the stationary distribution of Z is $T.x$ as the following lemma shows.

Lemma 4.1 *If $z \stackrel{\text{def}}{=} T.x$ then we have: $z\hat{P} = z$, $z > 0$ and $z1^T = 1$.*

Proof. For $1 \leq m \leq M$, the m^{th} entry of $z\hat{P}$ is equal to:

$$\begin{aligned}
& \sum_{l=1}^{M} z(l)\hat{P}(l,m) \\
= & \sum_{l=1}^{M}\Big(\sum_{i\in B(l)} x(i) \sum_{j\in B(m)} P(i,j)\Big) \\
= & \sum_{i=1}^{N} x(i) \sum_{j\in B(m)} P(i,j) \\
= & \sum_{j\in B(m)} \Big(\sum_{i=1}^{N} x(i)P(i,j)\Big) \\
= & \sum_{j\in B(m)} x(j) = z(m)
\end{aligned}$$

the remainder of the proof is obvious. □

The construction is analogous in the continuous time case. Let $X = (X_t)_{t\geq 0}$ be an homogeneous irreducible Markov process evolving in continuous time with transition rate matrix A where $A(i,i) \stackrel{\text{def}}{=} -\sum_{j\neq i} A(i,j)$. Let π denote the equilibrium probability distribution of X, that is: $\pi A = 0$, $\pi > 0$ and $\pi 1^T = 1$. As in the discrete case, we construct the pseudo-aggregated process $Z = (Z_t)_{t\geq 0}$ from X with respect to the partition $\mathcal{B}$ by giving its transition rate matrix $\hat{A}$:

$$\forall i, j \in F,\ \hat{A}(i,j) \stackrel{\text{def}}{=} \sum_{k\in B(i)} \frac{\pi(k)}{\sum_{h\in B(i)} \pi(h)} \sum_{l\in B(j)} A(k,l)$$

We have the same result about the stationary distribution of Z as in the discrete time case:

Lemma 4.2 *If $z \stackrel{\text{def}}{=} T.\pi$ then we have: $z\hat{A} = 0$, $z > 0$ and $z1^T = 1$.*

Proof. Since it is immediate that $z > 0$ and $z1^T = 1$, let us simply verify, as in Lemma 4.1, that $z\hat{A} = 0$. For $1 \leq m \leq M$, the m^{th} entry of $z\hat{A}$ is equal to:

$$\begin{aligned}
& \sum_{l=1}^{M} z(l)\hat{A}(l,m) \\
= & \sum_{l=1}^{M}\Big(\sum_{i\in B(l)} \pi(i) \sum_{j\in B(m)} A(i,j)\Big) \\
= & \sum_{i=1}^{N} \pi(i) \sum_{j\in B(m)} A(i,j) \\
= & \sum_{j\in B(m)} \Big(\sum_{i=1}^{N} \pi(i)A(i,j)\Big) \\
= & \ 0
\end{aligned}$$

wich concludes the proof. □

In both discrete and continuous time cases, we have the following property:

Lemma 4.3 *The pseudo-aggregated process constructed from X with respect to the partition $\mathcal{B}$ and the pseudo-aggregated process obtained after M successive aggregations of X with respect to each $B(i)$, $i \in F$, in any order, are equivalent.*

The proof is an immediate consequence of the construction of the pseudo-aggregated processes. It is for this reason that in the sequel we will consider only the situation where $\mathcal{B}$ contains only one subset having more than one state. That is, we will assume as in the previous sections, that $B = \{1, \ldots, L\}$ where $1 < L < N$ and that $\mathcal{B} = \{B, \{L+1\}, \ldots, \{N\}\}$ with $N \geq 3$. The state space of the pseudo-aggregated process Z will be denoted by $F = \{b, L+1, ..., N\}$. We will denote also by B^c the complementary subset $\{L+1, \ldots, N\}$ and by α the initial distribution of X.

4.2 Pseudo-aggregation and sojourn times: discrete time case

We consider now the pseudo-aggregated homogeneous Markov chain Z constructed from X with respect to the partition $\mathcal{B} = \{B, \{L+1\}, ..., \{N\}\}$ of E. Although the stationary distribution of X over the sets of $\mathcal{B}$ is equal to the state stationary distribution of Z (Lemma 4.1), it is not the same with the distribution of the n^{th} sojourn time of X in B and the corresponding distribution of $H_{b,n}$, the n^{th} holding time of Z in b, which is independent of n and will be then denoted by H_b. This last (geometric) distribution is given by:

$$P(H_b = k) = \frac{x_B(I - P_B)1^T}{x_B 1^T}\left(\frac{x_B P_B 1^T}{x_B 1^T}\right)^{k-1} \quad , \ k \geq 1$$

which is to be compared with the given expression of $P(H_{B,n} = k)$.

Observe that if there is no internal transition between different states inside B and if the geometric holding time distributions of X in the individual states of B have the same parameter, we have obviously identity between the differents distributions of $H_{B,n}$, $H_{B,\infty}$ and H_b. That is, if $P_B = \beta I$ with $0 \leq \beta < 1$, we have:

$$P(H_{B,n} = k) = P(H_{B,\infty} = k) = P(H_b = k) = (1-\beta)\beta^{k-1}, \quad k \geq 1$$

More generally, if X is a lumpable Markov chain [1] over the partition $\mathcal{B}$ (i.e. the aggregated chain Y is also Markov homogeneous for any initial distribution of X) we have: $P_B 1^T = \beta 1^T$, $(I - P_B)1^T = (1-\beta)1^T$, where $\beta \in [0,1[$. In this case, $P_B^{k-1}(I - P_B)1^T = (1-\beta)\beta^{k-1}$, $k \geq 1$, and the distributions of $H_{B,n}$, $H_{B,\infty}$ and H_b are identical.

We will investigate now the relation between moments (factorial moments) of $H_{B,n}$ and $H_{b,n}$. We give an algebraic necessary and sufficient condition for the equality between the Cesaro limit of the k-order moments of the random variable $H_{B,n}$ and the k-order moments of H_b.

Theorem 4.4 *For any $k \geq 1$:*

$$\left[\lim_{n\to+\infty} \frac{1}{n}\sum_{l=1}^{n} FM_k(H_{B,l}) = FM_k(H_b)\right]$$

$$\Longleftrightarrow$$

$$\left[x_B\left(P_B(I-P_B)^{-1}\right)^{k-1} 1^T = x_B 1^T\left(\frac{x_B P_B 1^T}{x_B(I-P_B)1^T}\right)^{k-1}\right]$$

Proof. For the geometric distribution of H_b, we have:

$$FM_k(H_b) = \frac{k!\left(x_B P_B 1^T\right)^{k-1} x_B 1^T}{(x_B(I-P_B)1^T)^k}$$

Let us fix $k \geq 1$. We have:

$$\left[\lim_{n\to+\infty} \frac{1}{n}\sum_{l=1}^{n} FM_k(H_{B,l}) = FM_k(H_b)\right]$$

$$\Longleftrightarrow \left[vP_B^{k-1}(I-P_B)^{-k}1^T = (x_B 1^T)\frac{\left(x_B P_B 1^T\right)^{k-1}}{(x_B(I-P_B)1^T)^k}\right]$$

$$\Longleftrightarrow \left[\left(x_B(I-P_B)1^T\right)^k vP_B^{k-1}(I-P_B)^{-k}1^T = (x_B 1^T)\left(x_B P_B 1^T\right)^{k-1}\right]$$

$$\Longleftrightarrow \left[\left(x_B(I-P_B)1^T\right)^{k-1} x_B(I-P_B)1^T v(I-P_B)^{-1}(I-P_B)^{-(k-1)}P_B^{k-1}1^T = (x_B 1^T)\left(x_B P_B 1^T\right)^{k-1}\right]$$

since $P_B(I-P_B)^{-1} = \sum_{j\geq 1} P_B^j = (I-P_B)^{-1}P_B$

$$\Longleftrightarrow \left[\left(x_B(I-P_B)1^T\right)^{k-1} x_B(I-P_B)\left(\lim_{n\to+\infty}\frac{1}{n}\sum_{l=1}^{n} G^l\right)(I-P_B)^{-1}(I-P_B)^{-(k-1)}P_B^{k-1}1^T = (x_B 1^T)\left(x_B P_B 1^T\right)^{k-1}\right]$$

$$\Longleftrightarrow \left[\left(x_B(I-P_B)1^T\right)^{k-1} x_B\left(\lim_{n\to+\infty}\frac{1}{n}\sum_{l=1}^{n}\left((I-P_B)G(I-P_B)^{-1}\right)^l\right)(I-P_B)^{-(k-1)}P_B^{k-1}1^T = (x_B 1^T)\left(x_B P_B 1^T\right)^{k-1}\right]$$

$$\Longleftrightarrow \left[\left(x_B(I-P_B)1^T\right)^{k-1} x_B\left(\lim_{n\to+\infty}\frac{1}{n}\sum_{l=1}^{n} U_B^l\right)(I-P_B)^{-(k-1)}P_B^{k-1}1^T = (x_B 1^T)\left(x_B P_B 1^T\right)^{k-1}\right]$$

$$\Longleftrightarrow \left[\left(x_B(I-P_B)1^T\right)^{k-1} x_B(I-P_B)^{-(k-1)}P_B^{k-1}1^T = (x_B 1^T)\left(x_B P_B 1^T\right)^{k-1}\right]$$

since $x_B U_B = x_B$ (see Section 2)

$$\Longleftrightarrow \left[x_B \left(P_B(I-P_B)^{-1}\right)^{k-1} 1^T = (x_B 1^T)\left(\frac{x_B P_B 1^T}{x_B(I-P_B)1^T}\right)^{k-1} \right]$$

which is the given condition □

In the important case $k = 1$, the above condition is always satisfied. The following corollary states this result. There is no need for a proof since it is trivial to check the condition for $k = 1$.

Corollary 4.5

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{l=1}^{n} E(H_{B,l}) = E(H_b)$$

Consider the numerical example given at the end of Section 2 where the distribution of $H_{B,n}$ is derived. When $n \to \infty$, the probabilities $P(H_{B,n} = h)$ converge simply (G' is aperiodic). The limits are:

$$P(H_{B,\infty} = 2h) = \frac{11}{5}\left(\frac{1}{8}\right)^h \quad h \geq 1$$

$$P(H_{B,\infty} = 2h+1) = \frac{3}{5}\left(\frac{1}{8}\right)^h \quad h \geq 0$$

In the pseudo-aggregated chain Z, the corresponding holding time distribution is:

$$P(H_b = h) = \frac{5}{8}\left(\frac{1}{8}\right)^{h-1} \quad h \geq 1$$

and we can verify that $E(H_{B,\infty}) = E(H_b) = \frac{8}{5}$.

When we consider greater order moments, the equality is no more valid as the following counterexample will show. We will compute here the standard moments of the involved random variables $H_{B,n}$ and H_b.

$E = \{1,2,3\}$, $B = \{1,2\}$ and

$$P = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ \hline 1 & 0 & 0 \end{array}\right)$$

The stationary probability vector is: $x = (2/5, 2/5, 1/5)$.
The expression giving the second moment of $H_{B,n}$ is:

$$E(H_{B,n}^2) = v_n(I + P_B)(I - P_B)^{-2} 1^T$$

Taking now limits in the sense of Cesaro, we obtain:

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{l=1}^{n}E(H_{B,l}^{2}) = v(I+P_B)(I-P_B)^{-2}1^T$$

which is equal to 24 in the example.
On the pseudo-aggregated process, we have:

$$E(H_b^2) = \frac{2-\rho}{\rho^2} \quad \text{where} \quad \rho = \frac{x_B(I-P_B)1^T}{x_B 1^T}$$

which is equal to 28 in the example.

4.3 Pseudo-aggregation and sojourn times: continuous time case

In continuous time we have analogous results about the relation between properties of sojourn times of the given process X and the corrresponding holding times of the pseudo-aggregated process Z.

Denote by H_b the holding time random variable for process Z.

$$P(H_b \le t) = 1 - e^{-\mu t} \qquad \text{where } \mu = -\frac{x_B\Lambda_B^{-1}A_B 1^T}{x_B\Lambda_B^{-1}1^T}$$

The continuous time version of Theorem 4.4 is then:

Theorem 4.6 *For any $k \ge 1$:*

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{l=1}^{n}E(H_{B,l}^{k}) = E(H_b^k)$$

$$\Longleftrightarrow$$

$$\left[(x_B(I-P_B)1^T)^{k-1}x_B\Lambda_B^{-1}((I-P_B)^{-1}\Lambda_B^{-1})^{k-1}1^T = (x_B\Lambda_B^{-1}1^T)^k\right]$$

Proof. The k-order moment of the holding time H_b of Z in b can be written:

$$E(H_b^k) = \frac{k!}{\mu^k} = k!\left(-\frac{x_B\Lambda_B^{-1}1^T}{x_B\Lambda_B^{-1}A_B 1^T}\right)^k$$

Then, we have:

$$\left[\lim_{n\to+\infty}\frac{1}{n}\sum_{l=1}^{n}E(H_{B,l}^{k}) = E(H_b^k)\right]$$

$$\Longleftrightarrow \left[v\left((I-P_B)^{-1}\Lambda_B^{-1}\right)^k 1^T = \left(\frac{x_B\Lambda_B^{-1}1^T}{x_B(I-P_B)1^T}\right)^k \right]$$

$$\Longleftrightarrow \left[x_B(I-P_B)1^T v\left((I-P_B)^{-1}\Lambda_B^{-1}\right)^k 1^T = \frac{\left(x_B\Lambda_B^{-1}1^T\right)^k}{\left(x_B(I-P_B)1^T\right)^{k-1}} \right]$$

$$\Longleftrightarrow \left[x_B(I-P_B)1^T v(I-P_B)^{-1}\Lambda_B^{-1}\left((I-P_B)^{-1}\Lambda_B^{-1}\right)^{k-1} 1^T \right.$$
$$\left. = \frac{\left(x_B\Lambda_B^{-1}1^T\right)^k}{\left(x_B(I-P_B)1^T\right)^{k-1}} \right]$$

$$\Longleftrightarrow \left[x_B\Lambda_B^{-1}\left((I-P_B)^{-1}\Lambda_B^{-1}\right)^{k-1} 1^T = \frac{\left(x_B\Lambda_B^{-1}1^T\right)^k}{\left(x_B(I-P_B)1^T\right)^{k-1}} \right]$$

$$\Longleftrightarrow \left[\left(x_B(I-P_B)1^T\right)^{k-1} x_B\Lambda_B^{-1}\left((I-P_B)^{-1}\Lambda_B^{-1}\right)^{k-1} 1^T = \left(x_B\Lambda_B^{-1}1^T\right)^k \right]$$

which concludes the proof □

As in the discrete time case, the given condition is always satisfied for first order moments. The following result corresponds to Corollary 4.5 with identical trivial verification.

Corollary 4.7

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{l=1}^{n} E(H_{B,l}) = E(H_{B,\infty}) = E(H_b)$$

5 Conclusions

In this paper we investigate the sojourn time of an homogeneous finite Markov process in a given subset of the state space in both the discrete and continuous time cases. In particular, analytical expressions of the distributions of the n^{th} sojourn or visit of the process to the subset are derived and their asymptotical behaviour is analyzed.

When the system analyst is interested only in steady state behaviour, it is usual to replace the original model by a "pseudo-aggregation" where the choosen subset is collapsed into a single state. This is done for instance when the state space has too many elements. This pseudo-aggregation is constructed such that the Markovian property is conserved even if the real aggregation of the original process is not Markov. The interest of this procedure is that the steady state probability that the original process will be in the given subset is equal to the steady state probability that the pseudo-aggregated one will be in the corresponding individual state. It is natural then to look at the relations between sojourn times in the first process and the corresponding holding times in the pseudo-aggregation. We show that we have equality between the limit in the sense of Cesaro of successive sojourn times expectations and the corresponding mean holding time in the associated pseudo-aggregated process and that the equality is no more valid when greater order moments are considered.

In addition to the natural possible extensions obtained by making different assumptions on the data, it will be of interest to investigate the application of this kind of result in areas such as reliability or performability.

References

[1] J.G. Kemeny and J.L Snell. *Finite Markov chains.* Springer-Verlag, New York Heidelberg Berlin, 1976.

[2] G. Rubino and B. Sericola. *On weak lumpability in Markov chains.* Technical Report 387, IRISA, Campus de Beaulieu, 35042 Rennes Cedex, France, 1987.

Imprimé en France
par
l'Institut National de Recherche en Informatique et en Automatique